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TWO EFFECTIVE METHODS OF SOLVING MIXED LINEAR PROBLEMS OF MECHANICS OF CONTINUOUS MEDIA

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One-dimensional integral equations of the first kind with irregular kernels containing a removable logarithmic singularity and a dimensionless, geometrical or physical parameter λ, are considered. The integral equations in question appear in the course of investigation of a wide class of mixed linear problems of the theory of elasticity and viscoelasticity (contact problems, problems on slits, inclusions and cover plates), hydrodynamics (linear problems of gliding, flows past slender profiles and surfaces, problems of linear supercavitation, etc.). All methods used in the past to study these integral equations are effective when the parameter λ is either large, or small [1]. This necessitated the use of more than one method, e.g. "the asymptotic method at large λ" and "the asymptotic method at small λ", to solve a single concrete problem. The present paper gives algorithm which are equally effective at all values of the parameter $\lambda \in (0, \infty]$.

1. Types of the integral equations of the mixed problems under investigation. Many mixed problems in the two- and three-dimensional formulation listed above can be reduced to an integral equation of the first kind of the convolution type in a finite interval

$$\int_{-1}^{1} \varphi(\xi) K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x) \quad (|x| \leqslant 1) \tag{1.1}$$

$$K(t) = \frac{1}{2} \int_{-\infty+ic}^{\infty+ic} \frac{L(u)}{u} e^{-iut} du \quad \left(t = \frac{x-\xi}{\lambda}, \quad u = \sigma + i\tau\right) \tag{1.2}$$

Using the properties of the function $L(u)$ we find, that in most cases the problems encountered can be divided into two classes [2]

$$\text{a) } L(u) = Au + O(u^2) \quad (u \to 0) \tag{1.3}$$
$$\text{b) } L(u) = (Bu)^{-1} + D^{-1} + O(u) \quad (u \to 0)$$

Moreover, in both cases we have, on the strip $|\tau| \leqslant \gamma, |\sigma| < \infty, |c| < \gamma$,

$$L(u) = 1 + O(e^{-\nu|\sigma|}) \quad (|\sigma| \to \infty) \tag{1.4}$$

and the function $L(u)$ is regular except at the point $u = 0$ in the case b). In (1.3) and (1.4) A, B, D, γ and ν are constants determined by the specific problems.

The restrictions imposed on the right-hand side of (1.1) will be shown below.

By virtue of the condition (1.3) we can write $L(u)$ for the case a) in the form

$$L(u) = \operatorname{th} Au + G_1(u) \tag{1.5}$$

Substituting (1.5) into (1.2) and integrating along the real axis we obtain, by virtue of (1.4) and the known integral [3]

$$-\ln\left|\operatorname{th}\frac{\pi y}{4}\right| = \int_0^\infty \frac{\operatorname{th} v}{v}\cos vy\, dv \tag{1.6}$$

the following expression for the kernel $K(t)$:

$$K(t) = -\ln\left|\operatorname{th}\frac{\pi t}{4A}\right| + N_1(t), \quad N_1(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{G_1(\sigma)}{\sigma}e^{-i\sigma t}\, d\sigma \tag{1.7}$$

On the basis of (1.4) and the regularity of the function $L(u)$ in the strip as well as the theorems A and B of [4], we can confirm that $N_1(t)$ is regular, as a function of the complex variable $w = t + is$, in the strip $|s| < \operatorname{Inf}(\nu, 2A)$, $|t| < \infty$.
We also have

$$N_1(t) = O(e^{-\varkappa|t|}) \quad \left(|t| \to \infty,\ \varkappa = \operatorname{Inf}\left(\gamma, \frac{\pi}{2A}\right)\right) \tag{1.8}$$

By virtue of the condition (1.3) for the case b), we can write the function $L(u)$ in the form

$$L(u) = \operatorname{cth} Bu + \frac{u}{\operatorname{sh} Du} + G_2(u) \tag{1.9}$$

Substituting (1.9) into (1.2) and integrating along the real axis, we use (1.4) and the relations [3]

$$-\ln\left|2\operatorname{sh}\frac{\pi y}{2}\right| + C = \int_0^\infty \frac{\operatorname{cth} v}{v}\cos vy\, dv \tag{1.10}$$

$$\frac{\pi}{2}\operatorname{th}\frac{\pi y}{2} = \int_0^\infty \frac{\sin vy}{\operatorname{sh} v}\, dv$$

taken in the sense of the theory of generalized functions [5] (C is an undefined constant) to obtain the following expression for the kernel $K(t)$:

$$K(t) = -\ln\left|2\operatorname{sh}\frac{\pi t}{2B}\right| + C - \frac{\pi i}{2D}\left(\operatorname{th}\frac{\pi t}{2D} \pm 1\right) + N_2(t) \tag{1.11}$$

$$N_2(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{G_2(\sigma)}{\sigma}e^{-i\sigma t}\, d\sigma \tag{1.12}$$

The plus and minus signs in (1.11) are chosen according to the position of the contour of integration in (1.2). As in the previous argument, we can assert that the function $N_2(t)$ is regular in the strip

$$|s| < \operatorname{Inf}(\nu, 2B, D), \quad |t| < \infty.$$

Moreover, when $|t| \to \infty$, the estimate (1.8) where $\varkappa = \operatorname{Inf}(\gamma, \pi/2B, \pi/D)$. holds for $N_2(t)$.

Thus the first terms in the expressions for $K(t)$ of the form (1.7) and (1.11) reflect fully all basic properties of the kernels of the integral equation (1.1), (1.2) for the cases a) and b) for all $t \in [0, \infty)$. When $t \in [0, \infty)$ the remaining terms in (1.7) and (1.11) are smooth as much as required. This implies that the exact inversion of the integral operators

$$L_a\varphi = -\int_{-1}^{1} \varphi(\xi)\ln\left|\operatorname{th}\frac{\pi t}{4A}\right|d\xi, \quad L_b\varphi = -\int_{-1}^{1} \varphi(\xi)\ln\left|2\operatorname{sh}\frac{\pi t}{2B}\right|d\xi \qquad (1.13)$$

will give a qualitatively exact description of the behavior of the solutions of (1.1), (1.2) for the cases a) and b) and for all values of the parameter λ. This can serve as the basis for developing an approximate method of solving the integral equations (1.1), (1.2) for both cases, which will be equally effective at all values of $\lambda \in (0, \infty]$. The idea of such an approach was outlined in [6].

Let us turn our attention to important auxiliary integral equations of the form

$$L_a\varphi = \pi g_1(x), \quad L_b\varphi = \pi g_2(x) \qquad (1.14)$$

We shall limit ourselves to the odd cases, i.e. we shall assume that the functions $g_1(x)$ and $g_2(x)$ and, therefore, the solutions of (1.14), are odd (the even cases have their own distinguishing features and require a separate consideration.

Let us perform in (1.14) a change of variables and introduce the corresponding notation:

$$\text{a)}\quad \beta = \frac{\operatorname{sh} r\xi}{\operatorname{sh} r}, \quad \alpha = \frac{\operatorname{sh} rx}{\operatorname{sh} r}, \quad r = \frac{\pi}{2A\lambda} \qquad (1.15)$$

$$\varphi^*(\beta) = \left(\frac{r\operatorname{ch} r\xi}{\operatorname{sh} r}\right)^{-1}\varphi(\xi), \quad g^*(\alpha) \equiv g_1(x)$$

$$\text{b)}\quad \beta = \frac{\operatorname{th} r\xi}{\operatorname{th} r}, \quad \alpha = \frac{\operatorname{th} rx}{\operatorname{th} r}, \quad r = \frac{\pi}{2B\lambda}$$

$$\varphi^*(\beta) = \left(\frac{r}{\operatorname{th} r\operatorname{ch}^2 r\xi}\right)^{-1}\varphi(\xi), \quad g^*(\alpha) \equiv g_2(x)$$

Equations (1.14) can now be written as a single expression

$$-\int_0^1 \varphi^*(\beta)\ln\left|\frac{\beta-\alpha}{\beta+\alpha}\right|d\beta = \pi g^*(\alpha) \quad (0 \leqslant \alpha \leqslant 1) \qquad (1.16)$$

Taking into account the fact that the functions $\varphi^*(\beta)$ and $g^*(\alpha)$ are odd, we can write (1.16) in the form

$$-\int_{-1}^{1} \varphi^*(\beta)\ln|\beta-\alpha|\,d\beta = \pi g^*(\alpha) \quad (|\alpha| \leqslant 1) \qquad (1.17)$$

Thus the problems of existence and uniqueness of the solution of (1.14) can be studied by investigating the same problems for (1.17).

2. On the structure of solution of the integral equations (1.14). We shall seek a solution of the integral equation (1.17) in the form

$$\varphi^*(\beta) = \omega(\beta)(1-\beta^2)^{-1/2} \qquad (2.1)$$

With regard to the function $\omega(\alpha)$ we shall assume that it belongs to the class $L_2^{1/2}(-1,1)$

which represents a complete space of functions with the norm

$$\|f\|^2 = \int_{-1}^{1} \frac{[f(\alpha)]^2}{\sqrt{1-\alpha^2}}\, d\alpha$$

Next we note that for the integral operator

$$M\omega = -\frac{1}{\pi}\int_{-1}^{1} \frac{\omega(\beta)}{\sqrt{1-\beta^2}} \ln|\beta-\alpha|\, d\beta$$

a system of eigenfunctions exists closed in $L_2^{1/2}(-1,1)$ which consists of the Tchebycheff polynomials of the first kind [7]

$$-\frac{1}{\pi}\int_{-1}^{1} \frac{T_n(\beta)}{\sqrt{1-\beta^2}} \ln|\beta-\alpha|\, d\beta = \frac{T_n(\alpha)}{c_n} \quad \left(c_0 = \frac{1}{\ln 2},\ c_n = n \geqslant 1\right) \tag{2.2}$$

The fact that the system is closed implies that any function $\omega(\alpha) \in L_2^{1/2}(-1, 1)$ can be uniquely represented by the expression [8]

$$\omega(\alpha) = \sum_{n=0}^{\infty} \omega_n T_n(\alpha) \quad (\|\omega\|_{L_2^{1/2}} = \|\omega\|_{l_2}) \tag{2.3}$$

where l_2 is a complete space of sequences with the norm

$$\|f\|^2 = \sum_{n=0}^{\infty} f_n^2 \quad (f = \{f_n\})$$

We assume that the function $g^*(\alpha)$ in (1.17) is such that $g^{*\prime}(\alpha) \in L_2^{1/2}(-1,1)$. Then the representation

$$g^*(\alpha) = \sum_{n=0}^{\infty} g_n T_n(\alpha) \tag{2.4}$$

will be even more possible for $g^*(\alpha)$. Substituting (2.1), (2.3) and (2.4) into (1.17) and using (2.2), we find that

$$\omega_n = c_n g_n \tag{2.5}$$

Theorem 1. If $g^{*\prime}(\alpha) \in L_2^{1/2}(-1, 1)$, then there exists a unique solution of the integral equation (1.17) such that $\varphi^*(\beta)$ has the form (2.1) and the function $\omega(\alpha) \in L_2^{1/2}(-1, 1)$. Moreover, the following correctness relation exists:

$$\|\omega(\alpha)\|^2_{L_2^{1/2}} \leqslant c_0^2 \left(\int_{-1}^{1} \frac{g^*(\alpha)\, d\alpha}{\sqrt{1-\alpha^2}}\right)^2 + m\|g^{*\prime}(\alpha)\|^2_{L_2^{1/2}} \quad (m = \text{const}) \tag{2.6}$$

which can also be written in the form

$$\|\varphi^*(\alpha)\|_{L_{4/3-0}} \leqslant m_1 \|g^*(\alpha)\|_{W^1_{4+0}} \quad (m_1 = \text{const}) \tag{2.7}$$

Here $L_p(-1,1)$ is a space of functions absolutely summable for $\alpha \in [-1, 1]$ with degree p, and $W_p^k(-1,1)$ is a space of functions the k-th derivatives of which are absolutely summable for $\alpha \in [-1, 1]$ with degree p.

To prove the theorem, we differentiate (2.4) in α. Taking into account (2.5) and the formulas

$$T_{2n}(\alpha) = 4n \sum_{k=1}^{n} T_{2k-1}(\alpha) \tag{2.8}$$

$$T'_{2n+1}(\alpha) = (2n+1)\left[T_0(\alpha) + 2 \sum_{k=1}^{n} T_{2k}(\alpha)\right]$$

we obtain

$$g^{*'}(\alpha) = \sum_{n=0}^{\infty} \frac{\omega_n}{c_n} T'_n(\alpha) = T_0(\alpha) \sum_{n=0}^{\infty} \omega_{2n+1} + 2 \sum_{k=1}^{\infty} T_{2k-1}(\alpha) \sum_{n=k}^{\infty} \omega_{2n} \tag{2.9}$$

$$+ 2 \sum_{k=1}^{\infty} T_{2k}(\alpha) \sum_{n=k}^{\infty} \omega_{2n+1}$$

On the other hand, the function $g^{*'}(\alpha)$ belonging to the class $L_2^{1/2}(-1,1)$, has the following expansion:

$$g^{*'}(\alpha) = \sum_{n=0}^{\infty} g'_n T_n(\alpha) \quad (\{g'_n\} \in l_2) \tag{2.10}$$

Comparing (2.9) with (2.10), we establish that

$$\omega_1 = g_0' - {}^1/_2\, g_2', \quad \omega_{2k+1} = g'_{2k} - g'_{2k+2}, \quad \omega_{2k} = g'_{2k-1} - g'_{2k+1}$$

and we can therefore write

$$\sum_{n=0}^{\infty} \omega_n^2 = c_0^2 g_0^2 + \left(g_0' - \frac{1}{2} g_2'\right)^2 + \sum_{k=1}^{\infty} [(g'_{2k} - g'_{2k+2})^2 \tag{2.11}$$

$$+ (g'_{2k-1} - g'_{2k+1})^2] \leqslant c_0^2 g_0^2 + m \sum_{n=0}^{\infty} g'^2_n$$

The above estimate uses the Cauchy – Buniakowski inequality. The relation (2.11) can also be written in the form

$$\| \omega(\alpha) \|_{l_2}^2 \leqslant c_0^2 g_0^2 + m \| g^{*'}(\alpha) \|_{l_2}^2$$

or, by virtue of equivalence of the norms in (2.3), as (2.6). Using the Hölder inequality, we can easily establish that

$$\| \varphi^*(\alpha) \|_{L_{4/3-0}} \leqslant \pi \| \omega(\alpha) \|_{L_2^{1/2}}, \qquad \| g^{*'}(\alpha) \|_{L_2^{1/2}} \leqslant m_2 \| g^{*'}(\alpha) \|_{L_{4+0}}$$

$(m_2 = \text{const})$

and thus confirm the validity of (2.7).

Corollary 1. From (2.7) follows the existence of a unique solution $\varphi^*(\alpha)$ of the integral equation (1.17) in the class $L_{4/3-0}(-1, 1)$ for $g^*(\alpha) \in W_{4+0}^1(-1,1)$.

When we use the results of (2.7), we must also remember that if $g^*(\alpha) \in W^1_{4+0}(-1,1)$, then $g^*(\alpha) \in B_0^{\mu}(-1,1)$, $0 < \mu \leqslant {}^3/_4$. The validity of the above statement can also be confirmed using the Hölder inequality. Here $B_k^{\mu}(-1,1)$ is the space of functions the k-th derivative of which satisfies, for $|\alpha| < 1$, the Hölder inequality with the index $0 < \mu \leqslant 1$. We also note that if $g^*(\alpha) \in B_1^{\mu}(-1,1)$ and $\mu > 0$, then, as was shown in [9], $\omega(\alpha) \in \beta_0^\nu(-1, 1)$ and $\nu = \mu$ when $\mu < 1$ and $\nu = 1-0$ when $\mu = 1$.

Using the facts proved for (1.17) we can now assert that for $g_i(x) \in W_{4+0}^1(-1, 1)$ $(i = 1, 2)$, unique solutions of the integral equations (1.14) exist of the form

$$\text{a)}\ \varphi(x) = \frac{\omega_1(x)\operatorname{ch} rx}{\sqrt{\operatorname{ch} 2r - \operatorname{ch} 2rx}}, \qquad \text{b)}\ \varphi(x) = \frac{\omega_2(x)}{\operatorname{ch} rx \sqrt{\operatorname{ch} 2r - \operatorname{ch} 2rx}} \quad (2.12)$$

where the functions $\omega_i(x) \in L_2^{1/2}(-1, 1)$ and the correctness relations (2.6) and (2.7) hold. If on the other hand $g_i(x) \in B_1^\mu(-1,1)$ and $\mu > 0$, then $\omega_i(x) \in B_0^\nu(-1,1)$ and $\nu = \mu$ for $\mu < 1$ and $\nu = 1-0$ for $\mu = 1$.

The following spectral relations will also be of use. They are obtained from (2.2) with (1.15) taken into account, and are

$$\text{a)}\ -\int_{-1}^{1} \frac{T_{2j+1}(\operatorname{sh} r\xi / \operatorname{sh} r)}{\sqrt{\operatorname{ch} 2r - \operatorname{ch} 2r\xi}} \ln\left|\operatorname{th}\frac{\pi(\xi - x)}{4A\lambda}\right| \operatorname{ch} r\xi\, d\xi = \pi\lambda_j T_{2j+1}\left(\frac{\operatorname{sh} rx}{\operatorname{sh} r}\right), \quad (2.13)$$

$$\lambda_j = [\sqrt{2}(2j+1)r]^{-1}$$

$$\text{b)}\ -\int_{-1}^{1} \frac{T_{2j+1}(\operatorname{th} r\xi / \operatorname{th} r)}{\sqrt{\operatorname{ch} 2r - \operatorname{ch} 2r\xi}} \ln\left|2\operatorname{sh}\frac{\pi(\xi - x)}{2B\lambda}\right| \frac{d\xi}{\operatorname{ch} r\xi} = \pi\lambda_j T_{2j+1}\left(\frac{\operatorname{th} rx}{\operatorname{th} r}\right),$$

$$\lambda_j = [\sqrt{2}(2j+1)r\operatorname{ch} r]^{-1}$$

Returning to the integral equation (1.1), (1.2) and taking into account the formulas (1.7) and (1.11), (1.12), we can rewrite it in the form

$$\text{a)}\ L_a\varphi = \pi f(x) - H_1\varphi, \qquad \text{b)}\ L_b\varphi = \pi f(x) - H_2\varphi \quad (2.14)$$

$$H_i\varphi = \int_{-1}^{1} \varphi(\xi) N_i\left(\frac{\xi - x}{\lambda}\right) d\xi \quad (i = 1, 2)$$

Here we have taken into account the fact that in the odd case of the variant b) the constant $D = \infty$. If we assume that $\varphi(x) \in L_{4/3-0}(-1,1)$, then the functions $H_i\varphi$ will be as smooth as required. This follows from the fact proved above that the function $N_i(t)$ is regular in some strip of the complex variable plane containing the real axis. Using Theorem 1, we can now formulate

Theorem 2. If the function $f(x) \in W_{4+0}^1(-1,1)$ and solutions of Eqs.(2.14) exist in the class $L_{4/3-0}(-1,1)$, then they have the form (2.12) for all values of the parameter $\lambda \in (0, \infty]$ and the functions $\omega_i(x) \in L_2^{1/2}(-1,1)$. At the same time, if $f(x) \in B_1^\mu(-1,1)$ and $\mu > 0$, then $\omega_i(x) \in B_0^\nu(-1,1)$ and $\nu = \mu(\mu < 1)$ and $\nu = 1-0$ $(\mu = 1)$.

3. Method of orthogonal polynomials. We shall seek the functions $\omega_i(\xi)(i = 1, 2)$ which appear in (2.12), in the form of the following series in Tchebycheff polynomials:

$$\omega_i(\xi) = \sum_{k=0}^{\infty} a_k T_{2k+1}(\beta) \quad (3.1)$$

By virtue of the properties of functions $\omega_i(\xi)$ shown in Theorem 2, the series (3.1) converge on the norm of the space $L_2^{1/2}(-1,1)$, and the corresponding sequences $\{a_k\}$ belong to the space l_2. Let us expand the function $f(x)$ and the regular supplements $N_i(t)$ $(i=1,2)$ to the kernels into the single and dual series respectively, in terms of the polynomial systems given above. We obtain

$$f(x)=\sum_{k=0}^{\infty} f_k T_{2k+1}(\alpha) \tag{3.2}$$
$$N_i(t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e_{mn}(\lambda) T_{2m+1}(\beta) T_{2n+1}(\alpha)$$

Here and henceforth the functions α and β will be given by the formulas (1.15) at $i=1$ and $i=2$ for the cases a) and b) respectively. Using the well known [3] property of orthogonality of the Tchebycheff polynomials, we obtain

$$e_{mn}(\lambda)=\frac{8r^2}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{N_i(t)\,T_{2m+1}(\beta)\,T_{2n+1}(\alpha)}{\sqrt{\operatorname{ch}2r-\operatorname{ch}2r\xi}\,\sqrt{\operatorname{ch}2r-\operatorname{ch}2rx}}\,q(\xi,x)\,d\xi\,dx \tag{3.3}$$

$$\text{a)}\quad q(\xi,x)=\operatorname{ch}r\xi\operatorname{ch}rx,\qquad \text{b)}\quad q(\xi,x)=\frac{\operatorname{ch}^2 r}{\operatorname{ch}r\xi\operatorname{ch}rx}$$

By virtue of the properties of the functions $f(x)$, $N_1(t)$ and $N_2(t)$ the series (3.2) converge uniformly [10] to those functions at all $|x|\leqslant 1$, $|\xi|\leqslant 1$ and $\lambda>0$.

Lemma 1. If the function $f(x)\in W^1_{4+0}(-1,1)$, then for any solution $\varphi(x)$ of class $L_{4|3-0}(-1,1)$ of the equation of the form (2.13) there exists a corresponding sequence of numbers a_i belonging to the class l_2 and satisfying the following infinite system of linear algebraic equations:

$$a_n=r_n-\sum_{m=0}^{\infty} a_m c_{mn}\quad (n=0,1,2,\ldots) \tag{3.4}$$
$$(r_n=f_n\lambda_n^{-1},\ c_{mn}={}^1\!/_2(2n+1)l_{mn})$$

Conversely, if the function $f(x)\in W^1_{4+0}(-1,1)$, then every solution $\{a_n\}$ of the system (3.4) belonging to the class l_2 has a corresponding solution $\varphi(x)\in L_{4/3-0}(-1,1)$ of the equation of the form (2.12), (3.1).

To prove this lemma, we take into account Theorem 2 and substitute into the integral equations (2.13) the functions $\varphi(x)$, $f(x)$, $N_1(t)$ and $N_2(t)$ in the form (2.12), (3.1) and (3.2). Then, using the spectral relations (2.13) and the property of orthogonality of the Tchebycheff polynomials we arrive, after certain manipulations, at (3.4). The converse procedure is equally easy.

Lemma 2. We have the following estimates for the coefficients $e_{mn}(\lambda)$ of the form (3.3):

$$\text{a)}\quad |e_{mn}(\lambda)|\leqslant\delta_n\frac{2\operatorname{sh}^3 r}{\pi^2 r^3\lambda^3(2m+1)}(2D_3+D_2\lambda r\operatorname{sh}r) \tag{3.5}$$
$$\text{b)}\quad |e_{mn}(\lambda)|\leqslant\delta_n\frac{\operatorname{sh}^3 2r}{2\pi^2\lambda^3 r^3(2m+1)}(D_3+D_2\lambda r\operatorname{th}r)$$

where $D_2 = \max|N_i''(t)|,\quad D_3 = \max|N_i'''(t)|\quad (|t| < \infty,\ i = 1, 2)$

$$\delta_n = \begin{cases} [n(n+1)]^{-1}, & n \geqslant 1 \\ \pi, & n = 0 \end{cases}$$

To prove this we perform in (3.3) the change of variables $\beta = \cos\psi$, $\alpha = \cos\varphi$. Then we integrate the resulting equations for $e_{mn}(\lambda)$ twice by parts in φ and once in ψ, and carry out a series of manipulations and estimates to arrive at (3.5).

Theorem 3. If the function $f(x) \in W^1_{4+0}(-1,1)$, then the operator appearing in the right-hand side of (3.4) acts in the space l_2, is completely continuous at all $\lambda \in (0,\infty]$ and is a contraction operator when $\lambda > \lambda_0$. The constant λ_0 is found from the equation

$$\text{a)}\ S_1(r) = \frac{1}{8}\left(\frac{4}{3} + \frac{1}{\pi^2}\right)\left[\frac{8A^3\operatorname{sh}^3 r}{\pi^3}\left(2D_3 + \frac{\pi}{2A}D_2\operatorname{sh} r\right)\right]^2 = 1 \qquad (3.6)$$

$$\text{b)}\ S_2(r) = \frac{1}{128}\left(\frac{4}{3} + \frac{1}{\pi^2}\right)\left[\frac{8B^3\operatorname{sh}^3 2r}{\pi^3}\left(D_3 + \frac{\pi}{2B}D_2\operatorname{th} r\right)\right]^2 = 1$$

To prove the theorem we change the variables in the formulas (3.2), according to (1.15). Differentiating the resulting relations with respect to α, taking into account (2.8) and remembering that $f'(x) \in L_2^{1/2}(-1,1)$, we confirm that the sequence $\{r_n\} \in l_2$.

Further, using the estimates (3.5) we can show that when $\lambda > 0$,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}^2 < S_i(r) < \infty \quad (i = 1, 2) \qquad (3.7)$$

From (3.7) it follows that the operator appearing in the right-hand side of (3.4) acts in the space of sequences l_2 and is completely continuous when $\lambda \in (0,\infty]$ [8]. Thus the infinite system (3.4) has a unique solution for almost every λ. From (3.7) we see that, when the relations (3.6) hold, the above operator will be a contraction operator in l_2. Consequently when $\lambda > \lambda_0$, a solution of the infinite system (3.4) exists in the space l_2, is unique and can be obtained with any degree of accuracy using the method of consecutive approximations or a reduction method [8]. We note that the infinite system (3.4) can also be written in the form

$$a_n^* = f_n - \sum_{m=0}^{\infty} a_m^* c_{mn}^* \qquad (3.8)$$

$$a_n^* = a_n\lambda_n, \quad c_{mn}^* = {}^1/_2\,(2m+1)\,e_{mn}(\lambda)$$

If the function $f(x) \in W^1_{4+0}(-1,1)$, it can be shown that $\{f_n\} \in l_1$ where l_1 is a complete space of sequences with the norm

$$\|f\| = \sum_{n=0}^{\infty}|f_n|$$

Having obtained the estimates of the type (3.5) for c_{mn}^*, we can also confirm that the operator appearing in the right-hand side of (3.8) acts in the space l_1. It can be proved that the infinite system (3.8) is quasi-completely regular when $\lambda > 0$. If it has a bounded solution, then $\{a_n^*\} \in l_1$. Some $\lambda_0^* > 0$ can be shown such that for $\lambda > \lambda_0^*$ the infinite system (3.8) is completely regular [11].

For the particular problem discussed in Sect. 5, we have $D_2 = 0.3466, D_3 = 0.1883$ and the relation (3.6) yields $\lambda_0 = 0.996$. Computations show that the reduction method converges for the system (3.4) also when $\lambda < \lambda_0$. It is important that the number of equations in the above system does not exceed, for the given degree of accuracy of solution, some value N for all $\lambda \in (0, \infty]$.

Having solved the system (3.4), we use the formulas (3.1) and (2.12) to find the solutions of the integral equations (2.14). The coefficient accompanying the singularity in the function $\varphi(x)$ can be found using the formula

$$\chi = \lim_{x\to 1} \varphi(x)\sqrt{1-x^2} = \frac{\sqrt{2r}}{\sqrt{\operatorname{sh} 2r}} \sum_{k=0}^{\infty} a_k$$

4. Method of collocation. After performing in the integral equations (2.14) the change of variables (1.15), we can write them in the form of a single expression

$$-\int_0^1 \varphi^*(\beta) \ln\left|\frac{\beta-\alpha}{\beta+\alpha}\right| d\beta = \pi f^*(\alpha) - \int_0^1 \varphi^*(\beta)\, m_k(\beta, \alpha, \lambda)\, d\beta \qquad (4.1)$$

$$m_k(\beta, \alpha, \lambda) \equiv N_k\left(\frac{\xi - x}{\lambda}\right) - N_k\left(\frac{\xi + x}{\lambda}\right) \quad (k = 1, 2;\ 0 \leqslant \alpha \leqslant 1)$$

We shall seek the solution of (4.1) in the form (2.1). Taking into account the relations $\beta = \cos\gamma$ and $\alpha = \cos\theta$ we obtain the following integral equation for $\omega(\cos\gamma)$:

$$-\int_0^{\pi/2} \omega(\cos\gamma) \ln\left|\frac{\cos\gamma - \cos\theta}{\cos\gamma + \cos\theta}\right| d\gamma = \qquad (4.2)$$

$$\pi f^*(\cos\theta) - \int_0^{\pi/2} \omega(\cos\gamma)\, m_k(\cos\gamma, \cos\theta, \lambda)\, d\gamma \quad \left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$$

Let us construct for the function $\omega(\cos\gamma)$ a Lagrange interpolation polynomial in terms of the Tchebycheff nodes [10]

$$\theta_s = \pi\frac{2s-1}{4i} \quad (s = 1, 2, \ldots, i)$$

Since the function $\omega(\cos\gamma)$ is odd, the polynomial will have the form

$$\omega(\cos\gamma) = \frac{2}{i}\sum_{s=1}^{i} \omega(\cos\theta_s) \sum_{n=1}^{i} \cos(2n-1)\theta_s \cos(2n-1)\gamma$$

Next we use the formula (2.2) to compute the integral in the left-hand side of (4.2). The integral appearing in the right-hand side of (4.2) can be computed using the Gauss quadrature formula [10]. Substituting the values of the integrals obtained into (4.2) and assigning to θ the following values:

$$\theta_j = \pi\frac{2j-1}{4i} \qquad (j = 1, 2, \ldots, i)$$

we obtain the following system of equations for $\omega(\cos\theta_s)$:

$$\frac{2}{i}\sum_{s=1}^{i}\omega(\cos\theta_s)\Big[\frac{1}{4}m_K(\cos\theta_s,\cos\theta_j,\lambda)+ \sum_{n=1}^{i}\cos(2n-1)\theta_s\frac{\cos(2n-1)\theta_j}{2n-1}=f^*(\cos\theta_j) \qquad (4.3)$$

Having solved the system (4.3), we obtain approximate solutions of (2.14) using the formulas

$$\varphi(x)=\frac{r\sqrt{2}\,\mathrm{ch}\,rx}{\sqrt{\mathrm{ch}\,2r-\mathrm{ch}\,2rx}}\Big[\frac{2}{i}\sum_{s=1}^{i}\omega(\cos\theta_s)\sum_{n=1}^{i}\cos(2n-1)\,\theta_s T_{2n-1}\Big(\frac{\mathrm{sh}\,rx}{\mathrm{sh}\,r}\Big)\Big]$$

$$\varphi(x)=\frac{r\sqrt{2}\,\mathrm{ch}\,r}{\mathrm{ch}\,rx\sqrt{\mathrm{ch}\,2r-\mathrm{ch}\,2rx}}\Big[\frac{2}{i}\sum_{s=1}^{i}\omega(\cos\theta_s)\sum_{n=1}^{i}\cos(2n-1)\,\theta_s T_{2n-1}\Big(\frac{\mathrm{th}\,rx}{\mathrm{th}\,r}\Big)\Big]$$

$$\chi=\lim_{x\to1}\varphi(x)\sqrt{1-x^2}=\sqrt{\frac{2r}{\mathrm{sh}\,2r}}\Big[\frac{2}{i}\sum_{s=1}^{i}\omega(\cos\theta_s)\sum_{n=1}^{i}\cos(2n-1)\theta_s\Big]$$

The convergence of the method with the increasing number of the collocation nodes can be proved with the help of the results of [12]. It must be noted that for the given accuracy of the approximate solution, the number i of nodes does not exceed some value i_0 for $\lambda\in(0,\infty]$.

5. Stretching of an elastic strip reinforced with a rigid cover plate of finite length. Let one of the boundaries of an elastic isotropic strip with elastic constants G and ν (G is shear modulus and ν is the Poisson's ratio) of thickness h, be connected to an inextensible, but perfectly elastic plate of length $2a$. We assume that outside the plate the strip boundary is stress - free. The opposite boundary of the strip lies without friction on a nondeformable support. We also assume that the strip is fully coupled to the plate within the region of their contact and, that the strip is stretched at infinity by the forces $P=ph$. We seek the tangential stresses $\tau(y)$ appearing in the region of contact between the strip and the plate.

The problem in question can reduced to that of solving an integral equation which, in the dimensionless coordinates, will have the form

$$\int_{-1}^{1}\varphi(\xi)K\Big(\frac{\xi-x}{\lambda}\Big)d\xi=-\pi x \quad (|x|\leqslant1) \qquad (5.1)$$

$$K\Big(\frac{\xi-x}{\lambda}\Big)=\int_0^\infty\frac{L(u)}{u}\cos\frac{\xi-x}{\lambda}u\,du$$

$$\tau(y)=\frac{p}{2}\varphi\Big(\frac{y}{a}\Big)\quad(|y|\leqslant a),\quad \lambda=\frac{h}{a},\quad L(u)=\frac{\mathrm{ch}\,2u+1}{\mathrm{sh}\,2u+2u}$$

and for the present problem we have, in (1.3), $B=2$, $D=\infty$.

An approximate solution of (5.1) for $\lambda>0$ can be obtained by one of the methods mentioned in Sect. 3 or 4. The necessary values of the function $N_2(t)$ were computed on a digital computer and are given in Table 1 ($N_2(t)\approx\exp(-{}^1/_2\pi t)$ for $t>4$)

Table 1

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$-N_2(t)\cdot 10^3$	415	413	405	394	379	360	339
t	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$-N_2(t)\cdot 10^3$	291	241	193	151	114	85	62
t	2.2	2.4	2.6	2.8	3.0	3.5	4.0
$-N_2(t)\cdot 10^3$	45	32	22	16	11	4	2

Table 2 gives the values of the function $\varphi(x)$ and of the coefficients accompanying the singularity χ, computed by both methods given in Sect. 3 and 4,

Table 2

λ	$\varphi(x)$					χ
	$x=0.1$	$x=0.3$	$x=0.5$	$x=0.7$	$x=0.9$	
2.0	—0.105	—0.321	—0.592	—2.997	—2.090	—1.002
	—0.104	—0.326	—0.596	—1.007	—2.107	—1.017
1.0	—0.114	—0.352	—0.624	—1.029	—2.062	—0.982
	—0.113	—0.350	—0.625	—1.023	—2.067	—0.980
0.5	—0.093	—0.283	—0.526	—0.857	—1.689	—0.792
	—0.092	—0.288	—0.524	—0.857	—1.690	—0.790

the first line giving the results obtained by the method of orthogonal polynomials and the second line using the method of collocation. To achieve the accuracy to within the second decimal place (in the worst case of $\lambda = 1/2$), we must take eight equations in the system (3.4) and seven equations in the system (4.3).

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